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Quantum motion with time-dependent disorder: the fluctuating-bond case

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Abstract. We determine the propagation properties of a quantum particle in a d-dimensional lattice with hopping disorder, delta correlated in time. The system is delocalized: the averaged transition probability shows a diffusive behaviour. Then, superimposed on the disorder, we consider a bias favouring the motion with a given orientation, as in the dynamics of flux lines in superconductors. The result is an effective Liouvillian for the density matrix, which is characterized by competition between single-particle and pair hopping. In this case the transition probability is determined in terms of excitonic motion, each exciton being extended along the bias direction. In the small-bias regime the hopping disorder is almost ineffective along the Bragg lines of the Brillouin zone, where drift dominates. Elsewhere the system undergoes diffusion. In the opposite regime we find the single-sided-hopping spectrum, as expected from the bias term, but, due to the hopping disorder, this undergoes an abrupt change of sign at the Bragg lines.

1. Introduction

Various studies have been devoted to quantum propagation in disordered lattices, including site and hopping disorder. Here we study time-dependent hopping: in general the adiabatic motion of a particle in a 'hot' background. The motion of a charge in a rapidly fluctuating effective magnetic field belongs to this class: here we give a lattice version of the problem. Time-dependent fluctuations of the magnetic field have been considered by Aronov and Wolfle in studying the behaviour of doped high-T_c materials, close to the metal–insulator transition [2]: their analysis was motivated by magnetoresistance measurements [3] on Bi 2:2:0:1 compounds. Tight-binding Hamiltonians were also considered for the dynamics of flux lines in superconductors, a widely investigated topic, both at the experimental and at the theoretical level: see, e.g., [5, 6] and references therein. Columnar defects, artificially produced by energetic heavy-ion radiation, have been used in order to pin flux lines and reduce dissipation. Greatly enhanced pinning has been obtained, e.g., in $YBa_2Cu_3O_7$ crystals with aligned columnar defects, produced by Sn-ion radiation [7, 8]. In the corresponding path integral description, the Euclidean time is the vortex line parameter and the horizontal coordinates of the columnar defects [9] define the lattice nodes. In a hollow cylindrical superconductor the longitudinal current creates a transverse magnetic field which forces the flux lines to tilt with respect to the vertical alignment. In the Hamiltonian this translates into a term that is linear in the momentum [10, 11], and anti-Hermitian. It can obviously be read as originating from an imaginary vector potential. This term explicitly breaks the space inversion symmetry: in fact the particle has different left- and right-hopping amplitudes (in a given direction). This is not to be confused with a chiral particle, characterized by single-sided but unitary propagation. Motion with a preferred orientation arises in various non-quantum-mechanical contexts, e.g. in population dynamics [12], in the transport of passive scalars in fluids [4], in directed percolation. The non-Hermitian hopping term has the effect of depinning the vortex lines, as shown by various authors: [10, 13, 14, 16]. Here we first consider time-dependent hopping with no bias. Due to the averaging, the appropriate object to be studied, rather than the wave function, is the density matrix. In terms of it, one reconstructs every transition probability.

Our approach relies on a second-quantization formalism, which proved to be very efficient in describing edge states in quantum Hall systems [1]. In the limit of fast fluctuations, memory effects are cancelled and the effective dynamics is described by a Liouvillian operator (see section 2). We find that the quantum particle undergoes classical diffusion. In section 3 we add the deterministic bias: the Liouvillian, in the second-quantization formalism, then takes the form of the so-called pair-hopping model Hamiltonian [17, 18]. Pure diffusion is now always frustrated; we find excitonic states, propagating with a non-trivial dispersion law. In section 4 we summarize our results and compare them with related work. In appendix A we derive a property for averages of time-ordered exponential operators; in appendix B we show that the transition probability, in the Hermitian case, can be obtained by resumming the ladder diagrams, i.e. that it coincides with the diffuson amplitude.

2. Disordered lattice

We start with a lattice Hamiltonian with time-dependent hopping disorder, including an anti-Hermitian term:

$$\hat{H}_0(t) = -\sum_{x,\mu}[(u(x,\mu;t) + w(x,\mu;t))|x\rangle\langle x+e_\mu| \\ + (u^*(x,\mu;t) - w^*(x,\mu;t))|x+e_\mu\rangle\langle x|]$$

where $\mu = 1, 2, \ldots, d$; d is the lattice dimension. We assume zero average Gaussian coefficients; the various amplitudes are mutually independent, with the correlators

$$\langle u(x,\mu;t)u^*(x',\mu';t')\rangle = \delta(t-t')\delta(x-x')\delta_{\mu,\mu'}D_u(x,\mu;t)$$
$$\langle w(x,\mu;t)w^*(x',\mu';t')\rangle = \delta(t-t')\delta(x-x')\delta_{\mu,\mu'}D_w(x,\mu;t).$$

As previously announced, we deal with the density operator in the second-quantization formalism. We introduce two, mutually commuting, Fermi (or equivalently Bose) operators $\hat{a}(x)$ and $\hat{b}(x)$, related to two independent copies of the system. Before averaging, they are associated with the retarded and advanced particle, and evolve independently with evolution operators $\hat{U}$ and $\hat{U}^*$. We define the operator $\hat{F}$ as the second-quantized evolution operator: its matrix elements in the (1 + 1)-particle sector are

$$\langle 0|\hat{b}(y)\hat{a}(x)\hat{F}(t,t')\hat{a}^+(x')\hat{b}^+(y')|0\rangle = \langle x,y|\hat{U}(t,t')\otimes\hat{U}^*(t,t')|x',y'\rangle. \tag{1}$$

$\hat{F}$ is associated with the following two-particle Hamiltonian:

$$\hat{H}(t) = -\sum_{x,\mu}[u(x,\mu;t)\hat{C}(x,\mu) + u^*(x,\mu;t)\hat{C}^+(x,\mu) \\ + w(x,\mu;t)\hat{B}(x,\mu) - w^*(x,\mu;t)\hat{B}^+(x,\mu)] \tag{2}$$

where

$$\hat{C}(x,\mu) = \hat{a}^+(x)\hat{a}(x+e_\mu) - \hat{b}^+(x+e_\mu)\hat{b}(x)$$
$$\hat{B}(x,\mu) = \hat{a}^+(x)\hat{a}(x+e_\mu) + \hat{b}^+(x+e_\mu)\hat{b}(x)$$
$$\hat{Q}(x) = \hat{a}^+(x)\hat{a}(x) - \hat{b}^+(x)\hat{b}(x).$$

Using the fact that the disorder is δ-correlated in time, one can exactly perform the averaging of the Neumann series and re-exponentiate the result, thus obtaining

$$\langle \hat{F}(t,t')\rangle = T\exp\left[-\mathrm{sgn}(t-t')\int_{t'}^{t}\mathrm{d}\tau\ \hat{H}_{eff}(\tau)\right]. \tag{3}$$

The effective Liouvillian $\hat{H}_{eff}$ has the form

$$\hat{H}_{eff}(t) = \frac{1}{2}\sum_{x,\mu}[D_u(x,\mu;t)\{\hat{C}(x,\mu),\hat{C}^+(x,\mu)\} - D_w(x,\mu;t)\{\hat{B}(x,\mu),\hat{B}^+(x,\mu)\}] \tag{4}$$

where $\{\,,\,\}$ denotes the anticommutator. The averaging has generated a quartic term, which couples the particle and the antiparticle. The non-linearity can easily be handled in this case: in fact $\hat{H}_{eff}$ can be written as a quantum spin Hamiltonian [1], if one starts with the fermionic representation. One verifies that an angular momentum algebra is obtained from $\hat{a}(x)$ and $\hat{b}(x)$. The angular momentum is given by

$$\begin{aligned}&\hat{J}^+(x) = \hat{a}^+(x)\hat{b}^+(x)\\ &2\hat{J}_3(x)+1 = \hat{a}^+(x)\hat{a}(x)+\hat{b}^+(x)\hat{b}(x) = \hat{N}(x).\end{aligned} \tag{5}$$

If we define $D_{\pm,x,\mu}(t) = D_u(x,\mu;t)\pm D_w(x,\mu;t)$, the Liouvillian turns into

$$\begin{aligned}\hat{H}_{eff}(t) = -\frac{1}{2}\sum_{x,\mu}&[4D_{+,x,\mu}(t)(\hat{J}_1(x)\hat{J}_1(x+e_\mu)+\hat{J}_2(x)\hat{J}_2(x+e_\mu))\\ &+D_{-,x,\mu}(t)(4\hat{J}_3(x)\hat{J}_3(x+e_\mu)+\hat{Q}(x)\hat{Q}(x+e_\mu)-1)].\end{aligned}$$

The planar term, which describes pair hopping, is ferromagnetic. The vertical term, which counts the particles, turns from ferromagnetic to antiferromagnetic as the anti-Hermitian disorder prevails over the Hermitian disorder. The angular momentum operators commute with the charge operators $\hat{Q}(x)$. Similarly the total number N_a of a-type particles, N_b, and $\hat{Q}(x)$ commute with the Hamiltonian. Obviously, as long as we are concerned with the density matrix, we are only involved in the $N_a = N_b = 1$ sector, as made explicit in the matrix elements written in equation (1). In a first class of eigenstates the particles are separated and do not propagate, since the hopping term acts only on doubly occupied sites, i.e. on pure states of the form $|x\rangle\langle x|$. Let us describe such eigenstates as localized. A second class is given by plane-wave superpositions (magnons) of doubly occupied sites (in the case of homogeneous disorder):

$$\hat{J}^+(p)|0\rangle = \frac{1}{(2\pi)^{d/2}}\sum_x \hat{J}^+(x)\exp(\mathrm{i}px)|0\rangle \qquad (|p_\mu|\leqslant\pi) \tag{6}$$

with eigenvalues

$$E(p;t) = 2\sum_\mu[D_{-,\mu}(t) - D_{+,\mu}(t)\cos p_\mu]. \tag{7}$$

The site transition probability is then decomposed into plane-wave contributions:

$$\langle|\langle x|\hat{U}(t,t')|x'\rangle|^2\rangle = \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi}\mathrm{d}p\ \exp\left[\mathrm{i}p(x-x')-\mathrm{sgn}(t-t')\int_{t'}^{t}\mathrm{d}\tau\ E(p;\tau)\right]. \tag{8}$$

The magnons are insensitive to any site potential: this rather unintuitive result depends on the delta correlation of the hopping amplitudes as shown in appendix C. Hence, with time-independent correlators and Hermitian disorder, the Liouvillian reduces to a lattice Laplacian: it is no surprise then that diffusive (long-range-order) modes are the outcome of averaging over fast time fluctuations. If the disorder has an anti-Hermitian part, hopping rates locally break space inversion invariance, and make some diffusive modes unstable. In fact, the spectrum of the Liouvillian is no longer positive definite: from the minus sign in the exponent in equation (3), one sees that the portion of the Brillouin zone inside the surface $E(p) = 0$ becomes unstable. Let us comment on the connection with the problem of a $d = 2$ particle in a magnetic field. In two dimensions, a fluctuating magnetic field orthogonal to the plane would be described by a hopping coefficient of the form $u(x, \mu, t) = \exp(\mathrm{i}\theta(x, \mu, t))$ with $\theta(x, \mu, t)$ Gaussian; in our model, u is instead Gaussian. In spite of this major difference, the present solution confirms a previous result on the motion in a fluctuating magnetic field, obtained in the continuum case [15]. The effective motion of the quantum particle is in both cases classical diffusion. We are not able to understand whether this is a mere coincidence or is related to some general property shared by the two approaches.

3. Biased system with disorder

We add now a deterministic asymmetric hopping, which describes a biased transport in a preferred direction [12]. The Hamiltonian, for the particle–antiparticle system, becomes

$$\begin{aligned}\hat{H}_{bias} = &\sum_{x,\mu} \delta_{\mu,\overline{\mu}}\alpha[\exp(-k)\hat{a}^{+}(x+e_\mu)\hat{a}(x) + \exp(k)\hat{a}^{+}(x)\hat{a}(x+e_\mu)] \\ &- \sum_{x,\mu} \delta_{\mu,\overline{\mu}}\alpha[\exp(-k)\hat{b}^{+}(x+e_\mu)\hat{b}(x) + \exp(k)\hat{b}^{+}(x)\hat{b}(x+e_\mu)]\end{aligned} \tag{9}$$

where k and α are real. Since in the perturbative series one can isolate the deterministic term and expand in the disorder term (see appendix A), the total effective Liouvillian $\hat{L}$ is simply $\hat{L} = \mathrm{i}\,\mathrm{sgn}(t-t')\hat{H}_{eff}(t) + \hat{H}_{bias}$, where $\hat{H}_{eff}$ is given in equation (4). If we consider homogeneous hopping disorder, we recover a non-Hermitian version of the so-called pair-hopping model Hamiltonian [18]. Notice that here the pair-hopping term is intrinsically dissipative. Neither the magnons nor the eigenstates of the bias term (free-particle states) are eigenstates. It is nonetheless possible to determine two families of solutions, which can be regarded as the natural extension of the previously determined ones (localized and diffusive, respectively). The wave function $f(x, y)$, in the two-particle sector ($N_a = N_b = 1$), satisfies the eigenvalue equation

$$\begin{aligned}&\alpha[\mathrm{e}^{-k} f(x-e_{\overline{\mu}}, y) + \mathrm{e}^{k} f(x+e_{\overline{\mu}}, y) - \mathrm{e}^{-k} f(x, y-e_{\overline{\mu}}) - \mathrm{e}^{k} f(x, y+e_{\overline{\mu}})] \\ &\quad + \mathrm{i}\sum_{\mu} D_{+}(\mu)\delta_{x,y}[f(x+e_\mu, y+e_\mu) + f(x-e_\mu, y-e_\mu)] - 2D_{-}(\mu) f(x, y) \\ &= \mathcal{E} f(x, y)\end{aligned} \tag{10}$$

where, in terms of the previous section's notation, we have $\mathcal{E} \equiv -\mathrm{i}E$. Since the pair-hopping term vanishes at singly occupied sites, two-particle states, separated in every other direction but the bias one, will only be acted on by the single-hopping term. This first class of solutions is the obvious extension of the formerly localized ones. In the second class there is no longer separation orthogonal to the bias. Let us proceed to the details of the solution. Upon writing f as a function of the baricentric coordinate $R = (x+y)/2$ and of the relative coordinate $r = x-y$, one easily identifies the eigenspace S^0, which can be spanned by the eigenfunctions:

$$f_{\mathcal{E},n,R^0}(R, r) = \exp(\mathrm{i}PR + \mathrm{i}q_{\overline{\mu}}r_{\overline{\mu}})\left[\prod_{\alpha,\mu\neq\overline{\mu}} \delta_{r_\alpha,n_\alpha}\delta_{R_\mu,R^0_\mu}\right] \tag{11}$$

where R^0, n are $(d-1)$-dimensional vectors, playing the role of degeneracy indexes. They are the projections of the baricentric and relative coordinates on the space $X_\perp$, orthogonal to the bias. The eigenvalues then, up to a constant, coincide with the eigenvalues of $\hat{H}_{bias}$:

$$\mathcal{E}(P_{\overline{\mu}}, q_{\overline{\mu}}) = -4\alpha \sin[(P/2)_{\overline{\mu}} - \mathrm{i}k] \sin[q_{\overline{\mu}}] - 2\mathrm{i} \sum_{\mu} D_-(\mu). \tag{12}$$

One verifies that S^0 is the eigenspace also upon adding a static disordered potential $V(x)$. The spectrum and eigenfunctions will then reproduce the features discussed by Hatano and Nelson [10]. To go over to the second class of solutions we first Fourier transform the equation:

$$\begin{aligned}(\mathcal{E} - 2\alpha \cos[(P/2-q)_{\overline{\mu}} - \mathrm{i}k] + 2\alpha \cos[(P/2+q)_{\overline{\mu}} - \mathrm{i}k]) f(P,q) \\ = -2\mathrm{i} \sum_{\mu} D_-(\mu) f(P,q) + 2\mathrm{i} \sum_{\mu} D_+(\mu) \cos(P_\mu) \frac{1}{(2\pi)^d} \int_{-\pi}^{+\pi} \mathrm{d}\overline{q}\; f(P,\overline{q}).\end{aligned} \tag{13}$$

After integrating over q, we get

$$(\mathrm{i}/2\pi) \int_{-\pi}^{+\pi} \frac{\mathrm{d}q'}{(\eta/z) + \sin(q')} = \frac{z}{4\xi} \tag{14}$$

where

$$\eta = (\mathcal{E}/4) + (\mathrm{i}/2) \sum_{\mu} D_-(\mu) \qquad z = \alpha \sin[P_{\overline{\mu}}/2 - \mathrm{i}k] \qquad \xi \equiv \frac{1}{2} \sum_{\mu} D_+(\mu) \cos(P_\mu)$$

where the integral is one dimensional. The eigenvalues are

$$\mathcal{E}(P) = -2\mathrm{i} \sum_{\mu} D_-(\mu) + 4\mathrm{i}\xi \left[1 - \left(\frac{z}{\xi}\right)^2\right]^{1/2}. \tag{15}$$

Notice that the solution is invariant under the symmetry $P \to -P, k \to -k$; of the two branches of the square root, we take the one that, as $\alpha \to 0$, reduces to the spectrum of the unbiased case (equation (7)). The wave function $f_{P_0}(P,q)$, associated with $\mathcal{E}(P_0)$, has the form

$$f_{P_0}(P,q) = \frac{\mathrm{i}\xi\delta(P-P_0)}{z \sin(q_{\overline{\mu}}) + \mathrm{i}\xi[1-(z/\xi)^2]^{1/2}}. \tag{16}$$

In the coordinate representation one finds an exponential behaviour: $f \approx \zeta_{\pm}^{r_{\overline{\mu}}}$, where

$$\zeta_{\pm} = \frac{\xi}{z} \left[\left[1 - \left(\frac{z}{\xi}\right)^2\right]^{1/2} \pm 1\right] \qquad (\zeta_+\zeta_- = -1) \tag{17}$$

and $\pm$ is to be chosen according to the condition $|\zeta_{\pm}| < 1$.

The function f is divergent as $r_{\overline{\mu}} \to -\infty$, but when computing the probability of transition between two sites one only needs $r_{\overline{\mu}} = 0$ both in f and in the solution of the transpose equation. We finally examine the case with $\zeta_{\pm}$ lying on the unit circle.

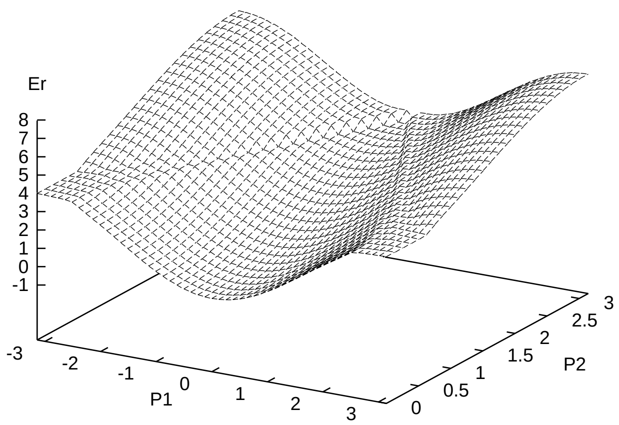
The condition $|\zeta_{\pm}| = 1$ implies ξ/z real, with absolute value smaller than one: this is true only in the absence of asymmetry in the deterministic term ($k = 0$), or when $P_{\overline{\mu}} = \pm\pi$. We have an exciton, extended along $\mu = \overline{\mu}$, with global momentum P_0 and relative momentum p_{eff}:

$$p_{eff} = \begin{cases} \arccos\left(\dfrac{\xi(P_0)}{\alpha \sin[(P_0)_{\overline{\mu}}/2]}\right) & (k=0) \\ \arccos\left(\dfrac{\xi(P_0)}{\alpha \cosh[k]}\right) & (P_{\overline{\mu}} = \pm\pi). \end{cases} \tag{18}$$

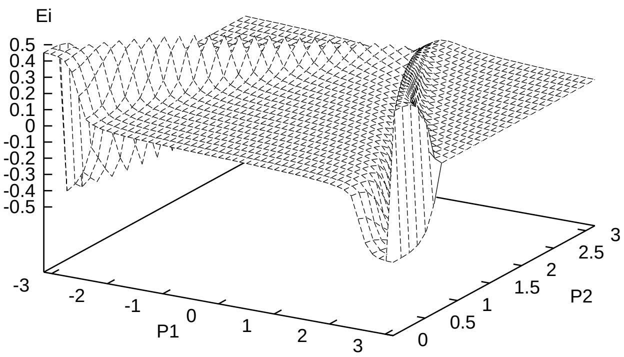
The spectrum of the Liouvillian $E(P)$ (equation (15)) for $d = 2$, with bias in the $\overline{\mu} = 1$ direction, and with Hermitian, isotropic disorder ($D_w(\mu) = 0$, $D_u(\mu) = D$), is exhibited in the figures. The imaginary part $E_I(P)$ of $E(P)$ describes the reversible motion, with drift velocity $v = \nabla_P E_I(P)$. The real part $E_R(P)$ describes the irreversible motion: for $k \neq 0$ a region of instability at the centre of the Brillouin zone appears. When α/D is small enough, i.e. when disorder dominates, drift is found along the Bragg lines $P_2 = \pm P_1 \pm \pi$ (see figure 1); among such lines, $E_I(P)$ is practically zero and the exciton predominantly undergoes diffusion. Upon increasing α, the bias exponent k tilts the plane $E_I(P)$ in the P_1-direction, thus enforcing drift at the centre of the Brillouin zone also. This is exhibited in figure 2. At very strong bias (k large) one would expect the single-sided hopping to dominate. The situation is rather different (see figure 3). One easily verifies that

$$E(P) \approx \mathrm{sgn}[(\cos(p_1) + \cos(p_2)](\alpha \exp(k))(\cos(p_1/2) + \mathrm{i} \sin(p_1/2)$$

where the last factor is indeed the eigenvalue of the bias operator in the single-sided-hopping limit. The effect of disorder is in the first factor: an abrupt change in sign of both the dissipative and reversible parts of the spectrum at the Bragg lines. Let us finally discuss the case $k = 0$, which describes a particle with deterministic anisotropic diffusion plus disorder. On qualitative grounds, everything goes as in figure 1, but now $E_R(P)$ is always positive (no instability



(a)



(b)

Figure 1. (a) The real part E_r of $E(P)$, over one half of the Brillouin zone ($-\pi < P_1 < \pi$, $0 < P_2 < \pi$), for the parameters $D = 1.0$, $k = 0.5$, $\alpha = 0.1$. (b) The imaginary part E_i of $E(P)$, for the same parameters as in (a).

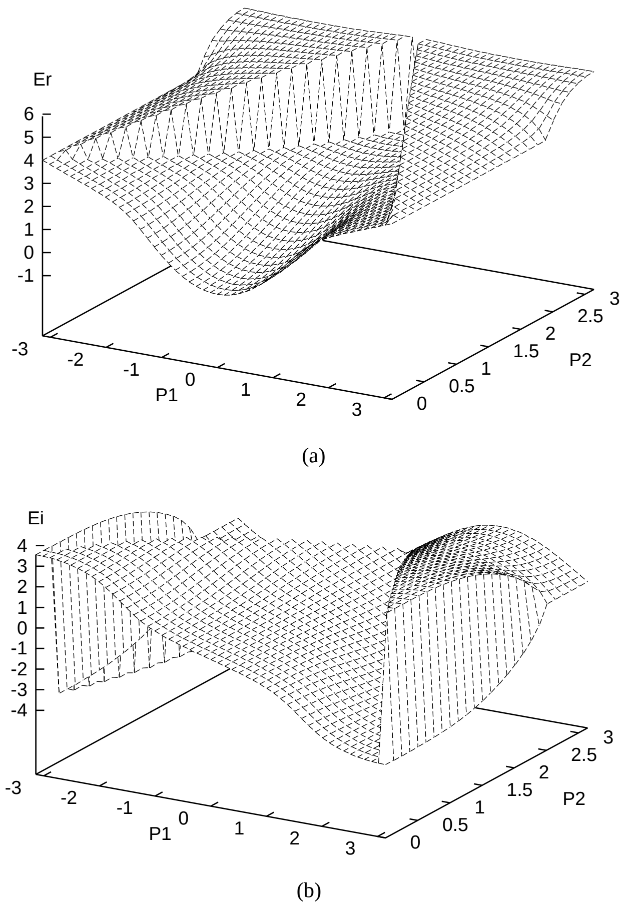
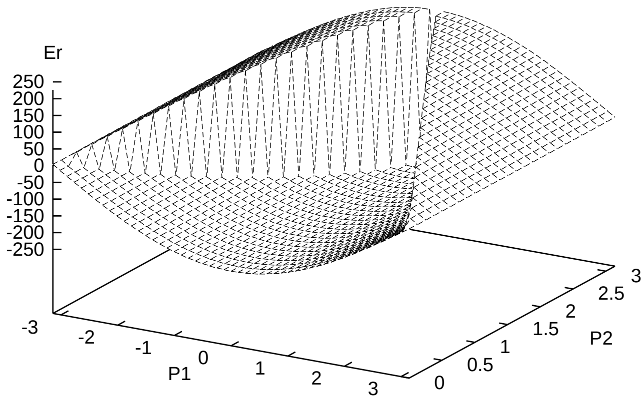


Figure 2. Competition between diffusion and bias: as for figure 1, but with $D = 1.0$, $k = 0.6$, $\alpha = 0.75$.

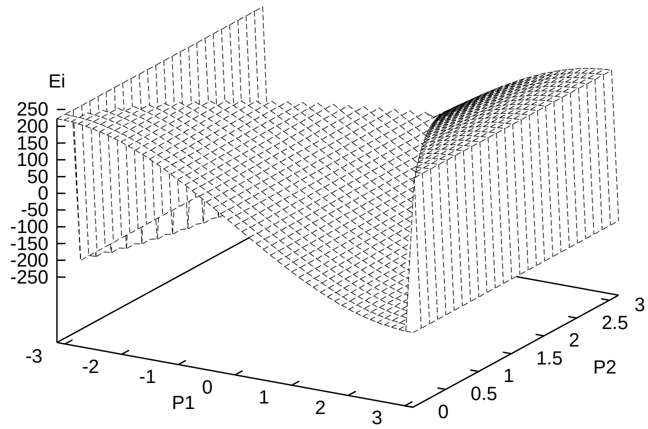
occurs). Reversible and irreversible motion are now completely separated, depending on P. Around the Bragg lines, we have only drift, apart from a constant damping factor; in the complementary region we find pure diffusion, with $E_I(P) = 0$ (no drift). The drift region tends to broaden as α is increased.

4. Conclusions

We have discussed the motion of a particle over a lattice with rapidly fluctuating hopping amplitudes: the model describes a massive quantum object coupled with a high temperature background. In our formalism the transition probability is written as a transition amplitude for a two-particle quantum system. This makes the operation of averaging the probability over classical fluctuations simpler. Before averaging, the two particles evolve independently, respectively forward and backward in time; after averaging, they interact and their motion becomes irreversible. Their effective Hamiltonian, which is simply the Liouvillian of the density matrix for the original system, has a quartic interaction, which can be readily put in the form of a Heisenberg Hamiltonian. We determined the steady states of the Liouvillian:



(a)



(b)

Figure 3. Strong bias: $D = 1.0$, $k = 5$, $\alpha = 0.75$.

with them, one computes the density matrix and any single-particle transition probability (see equations (1), (8)). A first class of steady states has the two particles physically separated and localized. The site transition probability depends on double occupancy states: it is a sum of plane waves, evolving in time with a diffusive law. The generic behaviour of such quantum systems is then diffusion, and this holds true even in the presence of a disordered potential, as shown in appendix C: since the hopping amplitudes are delta correlated in time, they destroy the phase coherence of the wave function. Quantum interference effects, essential for localization, are thus absent. Diffusion in a quantum-mechanical system was found in the Harper model at its critical point [19, 20]. The present result, obtained from a lattice model, confirms a previous one, on the motion of a particle in a rapidly fluctuating magnetic field, derived in the continuum case [15]. In section 3 we added a deterministic, anisotropic bias, enforcing a favoured orientation along a given direction; as already illustrated, this term arises quite naturally in describing tilted vortex motion in superconductors. The Liouvillian then takes the form of the so-called pair-hopping model Hamiltonian (i.e. it includes both single-particle and pair hopping, and the two terms do not commute). In our context the coupling constants are complex, since we are mixing reversible and irreversible motion. Two classes of steady states can be found. In the first class the two particles do not interact, provided that their wave packets do not overlap in the plane orthogonal to the bias. On such states the hopping disorder has practically no effect, and the interesting physics is the depinning transition, as described by Hatano and Nelson. In the second class the two particles form an exciton, extended along the bias direction. The dispersion law is a non-trivial function of the exciton momentum. For small enough bias, the Brillouin zone splits into a diffusion-dominated and a drift-dominated part, the latter lying around the Bragg lines. The site transition probability, which adds over the exciton contributions, is the sum of two parts: essentially reversible evolution around the Bragg lines, and irreversible diffusion with no drift in the complementary region. Notice that this is different from a mere sum of the two types of motion, since the separation involves different regions of momenta. For very large bias the dispersion law reduces to the one-way hopping form $\exp(\mathrm{i}p_1/2)$, but multiplied by the sign of $\cos(p_1)+\cos(p_2)$: this is the signature of the hopping disorder, which translates into a singular behaviour along the Bragg lines. It is seen then that a perturbative approach fails also in the extremal regimes.

Appendix A

We first point out here a relevant property of Gaussian averages of time-ordered exponential operators, holding for perturbations delta correlated in time. Let us consider the operator $\hat{H}(t) = \hat{H}_0(t) + \hat{V}(t)$, where the perturbation term is given through its correlator

$$\langle \hat{V}(t) \otimes \hat{V}(t')\rangle = \delta(t-t')\hat{A}(t).$$

One has, by definition,

$$T\exp\left[\int_{t'}^{t} \mathrm{d}\tau\ \hat{H}(\tau)\right] = \sum_{l=0,\,p=\pm}^{\infty} p^l \int_{t_{l+1}=t'} \left[\prod_{m=1}^{l} \mathrm{d}t_m\ \theta(p(t_m - t_{m+1}))\hat{H}(t_m)\right]\theta(p(t-t_1)).$$

The averaging leads to

$$\left\langle T\exp\left[\int_{t'}^{t} \mathrm{d}\tau\ \hat{H}(\tau)\right]\right\rangle = T\exp\left[\int_{t'}^{t} \mathrm{d}\tau\ [\hat{H}_0(\tau) + \mathrm{sgn}(t-t')\hat{V}_{eff}(\tau)]\right]$$

$$\langle \hat{V}(t)\hat{V}(t')\rangle = 2\delta(t-t')\hat{V}_{eff}(t).$$

Appendix B

The calculation of the averaged transition probability (equation (8)), in the Hermitian case, can be performed also by means of an exact resummation of ladder diagrams; the probability then coincides with the diffuson amplitude. The retarded and advanced Green's functions are

$$[-\mathrm{i}\,\partial_t \mp \mathrm{i}\eta + \hat{h}(t)]\hat{G}^{\pm}(t,t') = \delta(t-t')$$

$$\hat{G}^{\pm}(t,t') = \pm\mathrm{i}\theta(\pm(t-t'))\hat{U}(t,t')$$

$$\hat{G}^{+}(t,t') = [\hat{G}^{-}(t',t)]^{+}$$

$$|\langle x|\hat{U}(t,t')|x'\rangle|^2 = |\langle x|\hat{G}^{+}(t,t')|x'\rangle|^2 + |\langle x|\hat{G}^{-}(t,t')|x'\rangle|^2.$$

The averaging of the $\hat{U}$-operator is performed first; we obtain an effective single-particle generator $\hat{h}_{eff}(t)$:

$$\langle \hat{U}(t,t')\rangle = T\exp\left[-\mathrm{sgn}(t-t')\int_{t'}^{t} \mathrm{d}\tau\ \hat{h}_{eff}(\tau)\right]$$

$$\hat{h}_{eff}(t) = \sum_{\mu,x} D(\mu;t)|x\rangle\langle x|.$$

Let us consider now the perturbative expansion for the average

$$\langle x|\hat{G}^+(t,t')|x'\rangle\langle x'|\hat{G}^-(t',t)|x\rangle$$

where $\hat{h}(t)$ is the perturbation. One can show that the expansion can be written as the sum of particle–antiparticle diagrams where the 'free-propagator' lines are substituted for with the exact averaged Green's functions $\hat{G}^{\pm}_{av}$, and the contractions involve only particle and antiparticle disorder vertices. In fact, when contracting particle–particle vertices one is computing contributions to the averaged propagator. Due to causality and to delta correlation in time, crossed diagrams are zero; one is left with the sum of ladder diagrams, i.e. only the diffuson survives. The basic contribution to the ladder has the form

$$\begin{aligned}\langle t,x;\bar{t},\overline{x}|\hat{M}|t',x';\overline{t'},\overline{x'}\rangle &= \sum_{y,\overline{y}}\langle\langle x|\hat{h}(t)|y\rangle\langle\overline{y}|\hat{h}(t)|\overline{x}\rangle\rangle_{av}\langle y|\hat{G}^+_{av}(t,t')|x'\rangle\langle\overline{x'}|\hat{G}^-_{av}(\overline{t'},\bar{t})|\overline{y}\rangle\\ &= \delta(t-\bar{t})\delta_{x,\overline{x}}\delta_{x',\overline{x'}}\theta(t-t')\theta(t-\overline{t'})\sum_{\mu}D(\mu;t)[\delta_{x,x'-e_\mu}+\delta_{x,x'+e_\mu}]\\ &\quad\times\exp\left[-\sum_{\mu}\left(\int_{t'}^{t}\mathrm{d}\tau\ D(\mu;\tau)-\int_{\overline{t'}}^{t}\mathrm{d}\tau\ D(\mu;\tau)\right)\right].\end{aligned}$$

Let us introduce the following operator:

$$\langle t,x|\hat{P}|t',x'\rangle = \theta(t-t')\sum_{\mu}D(\mu;t)[\delta_{x,x'-e_\mu}+\delta_{x,x'+e_\mu}]\exp\left[-2\sum_{\mu}\int_{t'}^{t}\mathrm{d}\tau\ D(\mu;\tau)\right]$$

and then use the identity

$$\langle t,x;\bar{t},\overline{x}|\hat{M}^l|t',x';\overline{t'},\overline{x'}\rangle = \delta(t-\bar{t})\delta_{x,\overline{x}}\delta_{x',\overline{x'}}\langle t,x|\hat{P}^l|t',x'\rangle.$$

The diffuson amplitude $\Delta(t,x;t',x')$ is given by

$$\begin{aligned}\Delta(t,x;t',x') &= \int\mathrm{d}t_1\ \theta(t-t_1)\exp\left[-2\sum_{\mu}\int_{t_1}^{t}\mathrm{d}\tau\ D(\mu;\tau)\right]\langle t_1,x|\sum_{l=0}^{\infty}\hat{P}^l|t',x'\rangle\\ &= \theta(t-t')\langle x|T\exp\left[\int_{t'}^{t}\mathrm{d}\tau\ \left(\hat{S}(\tau)-2\sum_{\mu}D(\mu;\tau)\right)\right]|x'\rangle\end{aligned}$$

$$\hat{S}(t) = \sum_{x,\mu}D(\mu;t)[|x\rangle\langle x+e_\mu|+|x+e_\mu\rangle\langle x|].$$

In the momentum representation one recovers diffusion:

$$\Delta(t,x;t',x') = \frac{\theta(t-t')}{(2\pi)^d}\int_{-\pi}^{+\pi}\mathrm{d}k\ \exp\left[\mathrm{i}k(x-x')-2\sum_{\mu}\int_{t'}^{t}\mathrm{d}\tau\ D(\mu;\tau)(1-\cos k_\mu)\right].$$

Appendix C

Let us consider the interaction representation by taking the hopping term as a perturbation; we have the evolution operator $\hat{O}$:

$$\hat{O}(t,t_0) = \sum_{x}\exp\left[-\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\ V(x;\tau)\right]|x\rangle\langle x| \tag{C.1}$$

The transformed Hamiltonian $\hat{h}(t)$ is then:

$$\hat{O}^+(t,t_0)\hat{h}_0(t)\hat{O}(t,t_0) = \hat{h}(t).$$

One verifies that $\hat{h}(t)$ is obtained from $\hat{h}_0(t)$ through the substitution

$$\begin{aligned} u(x,\mu;t) &\to u(x,\mu;t)\exp\left[\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\ (V(x+e_\mu;\tau)-V(x;\tau))\right] \\ w(x,\mu;t) &\to w(x,\mu;t)\exp\left[\mathrm{i}\int_{t_0}^{t}\mathrm{d}\tau\ (V(x+e_\mu;\tau)-V(x;\tau))\right]. \end{aligned} \tag{C.2}$$

The invariance of correlators under this transformation is the origin of the independence of equation (8) of the potential. Notice further that the following identity holds in general:

$$\hat{U}_0(t,t') = \hat{O}(t,t_0)\hat{U}(t,t')\hat{O}^{+}(t',t_0) \tag{C.3}$$

where $\hat{U}_0$ and $\hat{U}$ are the evolution operators of the original Hamiltonian $\hat{h}_0$ (see equation (1)) and of $\hat{h}$ respectively.

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